

## **ASYMPTOTIC STABILITY OF NONLINEAR DELAY- DIFFERENCE SYSTEM WITH MULTIPLE DELAYS VIA MATRIX INEQUALITIES**

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### **Abstract**

In this paper, we obtain some criteria for determining the asymptotic stability of the zero solution of nonlinear delay-difference system with multiple delays in terms of certain matrix inequalities by using a discrete version of the Lyapunov second method.

### **1. Introduction**

We consider nonlinear delay-difference system with multiple delays of the form

$$x(k+1) = Ax(k) + \sum_{i=1}^m B_i x(k-h_i) + f(k, x(k), x(k-h_1), \dots, x(k-h_m)), \quad (1)$$

where  $x(k) \in \Omega \subseteq \mathbb{R}^n$ ,  $0 \leq h_1 \leq \dots \leq h_m$ ,  $m \geq 1$ ,  $A$  and  $B_i$ ,  $i = 0, 1, \dots, m$

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are given  $n \times n$  constant matrices,  $f(k, x(k), x(k - h_1), \dots, x(k - h_m))$  is a given  $n$ -vector nonlinear perturbation function satisfying  $f(k, 0, 0, \dots, 0) = 0$ .

## 2. Preliminaries

We assume that the  $n$ -vector function nonlinear perturbations are bounded and satisfy the following hypotheses, respectively:

$$0 \leq \frac{f_i(r_1) - f_i(r_2)}{r_1 - r_2} \leq l_i, \quad \forall r_1, r_2 \in \mathbb{R}, \text{ and } r_1 \neq r_2, \quad (2)$$

where  $l_i > 0$  are constants for  $i = 1, 2, \dots, n$ .

By assumption (2), we know that the functions  $f_i(\cdot)$  satisfy

$$|f_i(x_i)| \leq l_i |x_i|, \quad i = 1, 2, \dots, n. \quad (3)$$

**Fact 1.** For any positive scalar  $\varepsilon$  and vectors  $x$  and  $y$ , the following inequality holds:

$$x^T y + y^T x \leq \varepsilon x^T x + \varepsilon^{-1} y^T y.$$

**Lemma 2.1.** *The zero solution of difference system is asymptotic stability, if there exists a positive definite function  $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}^+$  such that*

$$\exists \beta > 0 : \Delta V(x(k)) = V(x(k+1)) - V(x(k)) \leq -\beta \|x(k)\|^2,$$

*along the solution of the system. In the case, the above condition holds for all  $x(k) \in V_\delta$ , we say that the zero solution is locally asymptotically stable.*

**Lemma 2.2.** *For any constant symmetric matrix  $M \in \mathbb{R}^{n \times n}$ ,  $M = M^T > 0$ , scalar  $s \in \mathbb{Z}^+ \setminus \{0\}$ , vector function  $W : [0, s] \rightarrow \mathbb{R}^n$ , we have*

$$s \sum_{i=0}^{s-1} (w^T(i) M w(i)) \geq \left( \sum_{i=0}^{s-1} w(i) \right)^T M \left( \sum_{i=0}^{s-1} w(i) \right).$$

### 3. Main Results

In this section, we present the main results of this paper, which provides a sufficient condition for the asymptotic stability of the zero solution of (1) in terms of certain matrix inequalities.

**Theorem 3.1.** *The zero solution of the nonlinear delay-difference system (1) is asymptotically stable, if there exists the symmetric positive definite matrices  $P, G_i, W_i, i = 1, 2, \dots, m$  and  $L_i = \text{diag}[l_{i1}, \dots, l_{in}] > 0, i = 0, 1, \dots, m$  satisfying the following matrix inequalities:*

$$\Psi = \begin{pmatrix} (0,0) & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & (1,1) & (1,2) & \dots & (1,m) & 0 & 0 & 0 & \dots & 0 \\ 0 & (2,1) & (2,2) & \dots & (2,m) & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (m,1) & (m,2) & \dots & (m,m) & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & (m+1, m+1) & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (m+2, m+2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (2m, 2m) \end{pmatrix} < 0, \quad (4)$$

where

$$(0, 0) = A^T P A - P + \sum_{i=1}^m (h_i G_i + W_i) + \varepsilon A^T P P A + \varepsilon_1 A^T A + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) L_0 P L_0,$$

$$(i, j) = \sum_{i=1}^m \sum_{j=1}^m B_i^T P B_j + (\varepsilon^{-1} + \varepsilon_2) \sum_{i=1}^m \sum_{j=1}^m B_i^T B_j + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) \sum_{i=1}^m \sum_{j=1}^m L_i P L_j$$

$$- \sum_{i=1}^m W_i, \quad \forall i = j = \{1, 2, \dots, m\},$$

$$(i, j) = \sum_{i=1}^m \sum_{j=1}^m B_i^T P B_j + (\varepsilon^{-1} + \varepsilon_2) \sum_{i=1}^m \sum_{j=1}^m B_i^T B_j, \quad \forall i \neq j = \{1, 2, \dots, m\},$$

$$(i, j) = - \sum_{i=1}^m \sum_{j=1}^m h_i G_j, \quad \forall i = j = \{m+1, m+2, \dots, 2m\}, \text{ and}$$

$$f = f(k, x(k), x(k-h_1), x(k-h_2), \dots, x(k-h_m)).$$

**Proof.** Consider the Lyapunov function  $V(y(k)) = V_1(y(k)) + V_2(y(k)) + V_3(y(k))$ , where

$$V_1(y(k)) = x^T(k)Px(k),$$

$$V_2(y(k)) = \sum_{i=1}^m \sum_{j=k-h_i+1}^k (h-k+i)x^T(j)G_i x(j),$$

$$V_3(y(k)) = \sum_{i=1}^m \sum_{j=k-h_i+1}^k x^T(j)W_i x(j),$$

$P, G_i, W_i, \quad i = 1, 2, \dots, m$  being symmetric positive definite solutions of (4) and  $y(k) = [x(k), x(k-h_1), \dots, x(k-h_m)]$ .

Then difference of  $V(y(k))$  along trajectory solution of (1) is given by

$$\Delta V(y(k)) = \Delta V_1(y(k)) + \Delta V_2(y(k)) + \Delta V_3(y(k)), \text{ where}$$

$$\Delta V_1(y(k)) = V_1(x(k+1)) - V_1(x(k))$$

$$\begin{aligned} &= [Ax(k) + \sum_{i=1}^m B_i x(k-h_i) + f]^T P [Ax(k) + \sum_{i=1}^m B_i x(k-h_i) + f] \\ &\quad - x^T(k)Px(k) \\ &= x^T(k)[A^T PA - P]x(k) \\ &\quad + \sum_{i=1}^m x^T(k)A^T P B_i x(k-h_i) + \sum_{i=1}^m x^T(k-h_i)B_i^T P A x(k) \\ &\quad + x^T(k)A^T P f + f^T P A x(k) \\ &\quad + \sum_{i=1}^m x^T(k-h_i)B_i^T P f + f^T P B_i x(k-h_i) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i)B_i^T P B_j x(k-h_j) + f^T P f, \end{aligned}$$

$$\begin{aligned}\Delta V_2(y(k)) &= \Delta \left( \sum_{i=1}^m \sum_{j=k-h_i}^{k-1} (h_i - k + j) x^T(j) G_i x(j) \right) \\ &= \sum_{i=1}^m h_i x^T(k) G_i x(k) - \sum_{i=1}^m \sum_{j=k-h_i}^{k-1} x^T(j) G_i x(j),\end{aligned}$$

and

$$\begin{aligned}\Delta V_3(y(k)) &= \Delta \left( \sum_{i=1}^m \sum_{j=k-h_i}^{k-1} x^T(j) W_i x(j) \right) \\ &= \sum_{i=1}^m x^T(k) W_i x(k) - \sum_{i=1}^m x^T(k-h_i) W_i x(k-h_i),\end{aligned}\quad (5)$$

where (3) and **Fact 2.1** are utilized in (5), respectively.

Note that

$$\begin{aligned}& \sum_{i=1}^m x^T(k) A^T P B_i x(k-h_i) + \sum_{i=1}^m x^T(k-h_i) B_i^T P A x(k) \\ & \leq \varepsilon x^T(k) A^T P P A x(k) + \sum_{i=1}^m \sum_{j=1}^m \varepsilon^{-1} x^T(k-h_i) B_i^T B_j x(k-h_j), \\ & x^T(k) A^T P f + f^T P A x(k) \\ & \leq \varepsilon_1 x^T(k) A^T A x(k) + \varepsilon_1^{-1} f^T P f, \\ & \sum_{i=1}^m x^T(k-h_i) B_i^T P f + \sum_{i=1}^m f^T P B_i x(k-h_i) \\ & \leq \varepsilon_2 \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i) B_i^T B_j x(k-h_j) + \varepsilon_2^{-1} f^T P f,\end{aligned}$$

and

$$f^T P f \leq \sum_{i=1}^m \sum_{j=1}^m x^T(k) L_i P L_j x(k) + \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i) L_i P L_j x(k-h_j),$$

$$\begin{aligned}\varepsilon_1^{-1} f^T P f &\leq \varepsilon_1^{-1} \sum_{i=1}^m \sum_{j=1}^m x^T(k) L_i P L_j x(k) + \varepsilon_1^{-1} \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i) L_i P L_j x(k-h_j), \\ \varepsilon_2^{-1} f^T P f &\leq \varepsilon_2^{-1} \sum_{i=1}^m \sum_{j=1}^m x^T(k) L_i P L_j x(k) + \varepsilon_2^{-1} \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i) L_i P L_j x(k-h_j),\end{aligned}$$

hence

$$\begin{aligned}\Delta V_1(y(k)) &\leq x^T(k) [A^T P A - P + \varepsilon A^T P P A + \varepsilon_1 A^T A + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) L_0 P L_0] x(k) \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i) [B_i^T P B_j + (\varepsilon^{-1} + \varepsilon_2) B_i^T B_j + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) \\ &\quad \times L_i P L_j] x(k-h_j).\end{aligned}$$

Then, we have

$$\begin{aligned}\Delta V(y(k)) &\leq x^T(k) [A^T P A - P + \sum_{i=1}^m h_i G_i + \sum_{i=1}^m W_i + \varepsilon A^T P P A + \varepsilon_1 A^T A \\ &\quad + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) L_0 P L_0] x(k) + \sum_{i=1}^m \sum_{j=1}^m x^T(k-h_i) [B_i^T P B_j \\ &\quad + (\varepsilon^{-1} + \varepsilon_2) B_i^T B_j + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) L_i P L_j - W_i] x(k-h_j) \\ &\quad - \sum_{i=1}^m \sum_{j=k-h_i}^{k-1} x^T(j) G_i x(j).\end{aligned}$$

Using **Lemma 2.2**, we obtain

$$\sum_{i=1}^m \sum_{j=k-h_i}^{k-1} x^T(j) G_i x(j) \geq \sum_{i=1}^m \left( \sum_{j=k-h_i}^{k-1} \left( \frac{1}{h_i} x(j) \right) \right)^T (h_i G_i) \left( \sum_{j=k-h_i}^{k-1} \left( \frac{1}{h_i} x(j) \right) \right).$$

From the above inequality, it follows that:

$$\Delta V(y(k)) \leq x^T(k) [A^T P A - P + \sum_{i=1}^m h_i G_i + \sum_{i=1}^m W_i + \varepsilon A^T P P A + \varepsilon_1 A^T A$$

$$\begin{aligned}
& + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1)L_0PL_0]x(k) \\
& + \sum_{i=1}^m \sum_{j=1}^m x^T(k - h_i)[B_i^T PB_j + (\varepsilon^{-1} + \varepsilon_2)B_i^T B_j \\
& + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1)L_iPL_j - W_i]x(k - h_j) \\
& - \sum_{i=1}^m \left( \frac{1}{h_i} \sum_{j=k-h_i}^{k-1} x(j) \right)^T (h_i G_i) \left( \frac{1}{h_i} \sum_{j=k-h_i}^{k-1} x(j) \right) \\
& = \left( x^T(k), x^T(k - h_1), x^T(k - h_2), \dots, x^T(k - h_m), \right. \\
& \quad \left. \left( \frac{1}{h_1} \sum_{i=k-h_1}^{k-1} x(j) \right)^T, \left( \frac{1}{h_2} \sum_{i=k-h_2}^{k-1} x(j) \right)^T, \dots, \left( \frac{1}{h_m} \sum_{i=k-h_m}^{k-1} x(j) \right)^T \right) \\
& \times \begin{pmatrix} (0, 0) & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & (1, 1) & (1, 2) & \dots & (1, m) & 0 & 0 & 0 & \dots & 0 \\ 0 & (2, 1) & (2, 2) & \dots & (2, m) & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (m, 1) & (m, 2) & \dots & (m, m) & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & (m+1, m+1) & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (m+2, m+2) & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & (2m, 2m) \end{pmatrix} \\
& \times \begin{pmatrix} x(k) \\ x(k - h_1) \\ x(k - h_2) \\ \vdots \\ x(k - h_m) \\ \left( \frac{1}{h_1} \sum_{j=k-h_1}^{k-1} x(j) \right) \\ \left( \frac{1}{h_2} \sum_{j=k-h_2}^{k-1} x(j) \right) \\ \vdots \\ \left( \frac{1}{h_m} \sum_{j=k-h_m}^{k-1} x(j) \right) \end{pmatrix}
\end{aligned}$$

$$= y^T(k)\Psi y(k),$$

where

$$(0, 0) = A^T P A - P + \sum_{i=1}^m (h_i G_i + W_i) + \varepsilon A^T P P A + \varepsilon_1 A^T A + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) L_0 P L_0,$$

$$(i, j) = \sum_{i=1}^m \sum_{j=1}^m B_i^T P B_j + (\varepsilon^{-1} + \varepsilon_2) \sum_{i=1}^m \sum_{j=1}^m B_i^T B_j + (\varepsilon_1^{-1} + \varepsilon_2^{-1} + 1) \sum_{i=1}^m \sum_{j=1}^m L_i P L_j$$

$$- \sum_{i=1}^m W_i, \forall i = j = \{1, 2, \dots, m\},$$

$$(i, j) = \sum_{i=1}^m \sum_{j=1}^m B_i^T P B_j + (\varepsilon^{-1} + \varepsilon_2) \sum_{i=1}^m \sum_{j=1}^m B_i^T B_j, \forall i \neq j = \{1, 2, \dots, m\},$$

$$(i, j) = - \sum_{i=1}^m \sum_{j=1}^m h_i G_j, \forall i = j = \{m+1, m+2, \dots, 2m\}, \text{ and}$$

$$f = f(k, x(k), x(k-h_1), x(k-h_2), \dots, x(k-h_m)).$$

By the condition (4),  $\Delta V(y(k))$  is negative definite, namely, there is a number  $\beta > 0$  such that  $\Delta V(y(k)) \leq -\beta \|y(k)\|^2$ , and hence, the asymptotic stability of the system immediately follows from **Lemma 2.1**. This completes the proof.  $\square$

**Remark 3.1.** Theorem 3.1 gives a sufficient condition for the asymptotic stability of nonlinear delay-difference system (1) via matrix inequalities. These conditions are described in terms of certain diagonal matrix inequalities, which can be realized by using the linear matrix inequality algorithm proposed in [4]. According to F.D. Chen, X.Y. Liao, H.Y. Zhu, these conditions are described in terms of certain symmetric matrix inequalities, which can be realized by using the Schur complement lemma and linear matrix inequality algorithm proposed in [4].



## 5. Conclusions

In this paper, based on a discrete analogue of the Lyapunov second method, we have established a sufficient condition for the asymptotic stability of nonlinear delay-difference system with multiple delays in terms of certain matrix inequalities.

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